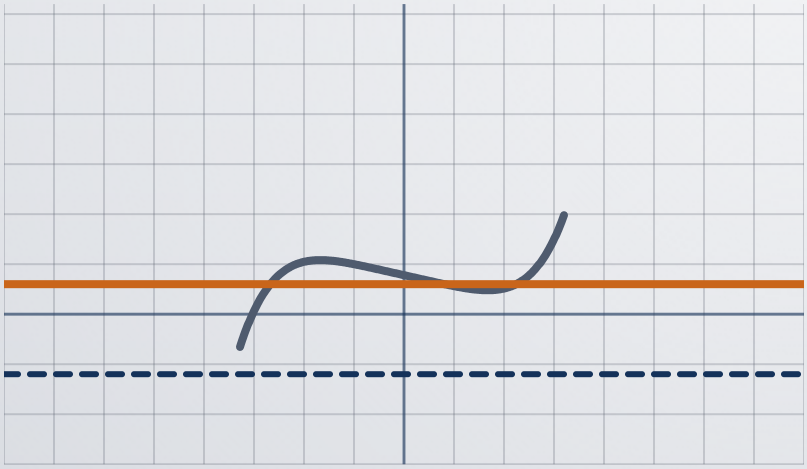
KECUDO

2026 EDITION · COLLEGE MATH SERIES



ABSTRACT ALGEBRA

Class J3

Fields & Galois Theory

Why the quintic has no formula

PREMIUM EDITION · KECUDO · J3

13 CHAPTERS · 120 SELF-ASSESSMENT QUESTIONS

INTERACTIVE EDITION + PRINT PDF

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Your Results **45**

●○○ FOUNDATION

Chapter 1

Fields

DEFINITION

A **field** is a commutative ring with $1 \neq 0$ in which every nonzero element has a multiplicative inverse: you can add, subtract, multiply and divide.

Field	Characteristic	Size
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	0	infinite
$\mathbb{Z}_p$ (p prime)	p	p
$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$	0	infinite
GF(4), GF(8), GF(2^n)	2	2^n

WORKED EXAMPLE **Inverse in $\mathbb{Q}(\sqrt{2})$**

1. $1/(1 + \sqrt{2}) = (\sqrt{2} - 1)/((\sqrt{2} + 1)(\sqrt{2} - 1)) = \sqrt{2} - 1$.

✓ **Rationalize with the conjugate**

PRIME SUBFIELD

Every field contains $\mathbb{Q}$ (characteristic 0) or $\mathbb{Z}_p$ (characteristic p).

●○○ FOUNDATION

Chapter 2

Extension Fields

E is an **extension** of F (written E/F) if F ⊆ E are both fields. F(α) is the smallest field containing F and α.

KRONECKER'S THEOREM

Every non-constant polynomial over F has a root in some extension: if p(x) is irreducible, E = F[x]/(p(x)) is a field in which x is a root.

WORKED EXAMPLE **Building ℂ and GF(4)**

1. $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$: the class of x is i.
2. $\mathbb{Z}_2[x]/(x^2 + x + 1)$ = GF(4) = {0, 1, α, α + 1} with $\alpha^2 = \alpha + 1$.

✓ Adjoin a root by quotienting

GF(4)'S MULTIPLICATION TABLE

×	1	α	α + 1
1	1	α	α + 1
α	α	α + 1	1
α + 1	α + 1	1	α

CHECK YOUR UNDERSTANDING

Chapter 1: Fields

5 questions · your answer locks when you choose it

1. Is $\mathbb{Z}_6$ a field?

A Only under +
B Only for odd elements
C No
D Yes

2. $1/(1 + \sqrt{2})$ in $\mathbb{Q}(\sqrt{2})$:

A $\sqrt{2} - 1$
B $1 - \sqrt{2}$
C $\sqrt{2}/2$
D $1 + \sqrt{2}$

3. Characteristic of GF(8):

A 0
B 2
C 8
D 3

4. The prime subfield of $\mathbb{R}$ is:

A $\mathbb{Z}_2$
B $\mathbb{R}$
C $\mathbb{Q}$
D $\mathbb{Z}$

5. Inverse of 3 in $\mathbb{Z}_7$:

A 3
B 4
C 5
D 2

FINAL EXAM

25 Questions on the Whole Book

Parts A–E · 5 questions each

Part A · Basics

1. A field allows division by:

A only 1 | **B** no element

C every nonzero element | **D** primes

2. [E : F] is:

A the characteristic | **B** the dimension of E over F

C the number of elements | **D** |F|

3. A splitting field is the smallest field where f:

A is constant | **B** has one root

C is irreducible | **D** factors into linear factors

4. The Galois group consists of automorphisms that fix:

A the roots | **B** E

C nothing | **D** the base field F

5. Galois: solvable by radicals ⇔ Galois group is:

A abelian | **B** solvable

C trivial | **D** simple



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Thank you for learning with Kecudo.

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