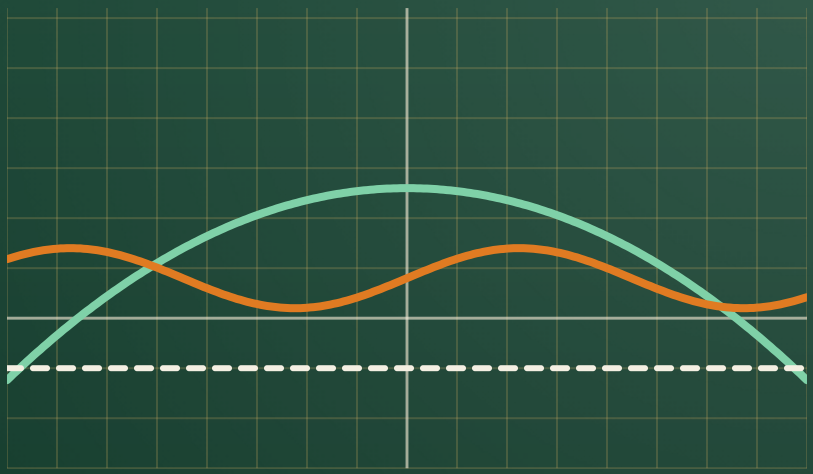
KECUDO

2026 EDITION · COLLEGE MATH SERIES



CALCULUS III

Class D3

Multiple Integrals

Adding up over areas and volumes

PREMIUM EDITION · KECUDO · D3

13 CHAPTERS · 120 SELF-ASSESSMENT QUESTIONS

INTERACTIVE EDITION + PRINT PDF

Contents

Part I · Foundations 4

1. Double Integrals over Rectangles 5
2. Iterated Integrals 8
3. General Regions 10
4. Changing the Order of Integration 13

Part I Review · 10 questions **16**

Part II · Core 18

5. Double Integrals in Polar 19
6. Mass & Center of Mass 22
7. Surface Area 25
8. Triple Integrals 27
9. Cylindrical Coordinates 29

Part II Review · 10 questions **31**

Part III · Advanced 33

10. Spherical Coordinates 34
11. Change of Variables & the Jacobian 37
12. Probability with Double Integrals 40
13. Moments of Inertia 43

Part III Review · 10 questions **45**

Final Exam · 25 questions **47**

Your Results **52**

FOUNDATION

Chapter 1

Double Integrals over Rectangles

Cut the rectangle R into small pieces of area ΔA, multiply each by the height $f(x_i^*, y_j^*)$, and add. As the pieces shrink, the sum approaches the **double integral**:

$$\iint_R f(x, y)\, dA = \lim \Sigma\, \Sigma\, f(x_i^*, y_j^*)\, \Delta A$$

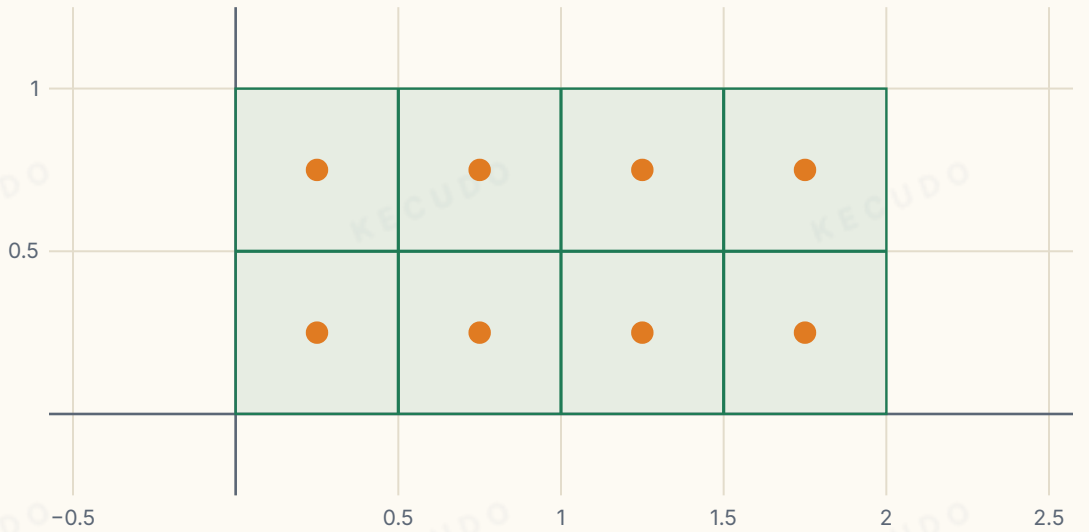


FIG 1.1 $R = [0, 2] \times [0, 1]$ cut into 8 squares, with a sample point (midpoint) in each.

WORKED EXAMPLE **Midpoint estimate of $\iint_R (x + 2y)\, dA$ with the 8 squares**

1. Each square has $\Delta A = 0.25$; the midpoint heights add to 16.
2. Estimate: $16(0.25) = 4$.

✓ **Exactly right here, because $x + 2y$ is linear**

MEANING

If $f \geq 0$, the double integral is the **volume** under the surface $z = f(x, y)$ and above R. Dividing by the area of R gives the **average value** of f.

WORKED EXAMPLE **Average height of $x + 2y$ over R**

1. $4/(\text{area } 2) = 2$.

✓ **Average value 2**

FOUNDATION

Chapter 2

Iterated Integrals

FUBINI'S THEOREM

For continuous f on $R = [a, b] \times [c, d]$:

$\iint_R f \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$.

Do the inner integral first, treating the outer variable as a constant: it's partial differentiation in reverse.

WORKED EXAMPLE $\int_0^2 \int_0^1 (x + 2y) \, dy \, dx$

1. Inner: $[xy + y^2]_0^1 = x + 1$.
2. Outer: $\int_0^2 (x + 1) \, dx = 2 + 2$.

✓ **4, matching Chapter 1**

WORKED EXAMPLE $\int_0^1 \int_0^2 x^2 y \, dx \, dy$

1. Inner: $(8/3)y$. Outer: $(8/3)(1/2)$.

✓ **4/3**

PRODUCT SHORTCUT

If $f(x, y) = g(x)h(y)$ on a rectangle, the double integral is $(\int g \, dx)(\int h \, dy)$.

CHECK YOUR UNDERSTANDING

Chapter 1: Double Integrals over Rectangles

5 questions · your answer locks when you choose it

1. For $f \geq 0$, $\iint_R f \, dA$ is:

A the average of f | **B** the slope

C the area of R | **D** the volume under $z = f$ over R

2. What is $\iint_R 3 \, dA$ over $R = [0, 2] \times [0, 5]$?

A 30 | **B** 10

C 3 | **D** 15

3. What is the area of $R = [1, 4] \times [0, 2]$?

A 3 | **B** 6

C 5 | **D** 8

4. If $\iint_R f \, dA = 12$ and R has area 4, what is the average value?

A 8 | **B** 16

C 3 | **D** 48

5. Each term of the Riemann sum is:

A ΔA | **B** f only

C f(sample point) $\times \Delta A$ | **D** $\Delta x + \Delta y$

FINAL EXAM

25 Questions on the Whole Book

Parts A–E · 5 questions each

Part A · Basics

1. For $f \geq 0$, $\iint_R f \, dA$ represents:

A volume under the surface
B arc length
C area of R
D slope

2. Fubini's theorem lets you:

A switch the order on a rectangle
B skip the inner integral
C ignore limits
D differentiate

3. In polar coordinates, $dA =$

A $r^2 \, dr \, d\theta$
B $dr \, d\theta$
C $\rho \, d\rho$
D $r \, dr \, d\theta$

4. In spherical coordinates, $dV =$

A $r \, dz \, dr \, d\theta$
B $\rho \, d\rho \, d\varphi$
C $d\rho \, d\theta \, d\varphi$
D $\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$

5. The Jacobian measures:

A probability
B curvature
C area or volume scaling in a change of variables
D density

K

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Thank you for learning with Kecudo.

NEXT IN THE SERIES

Class D4 · Vector Calculus