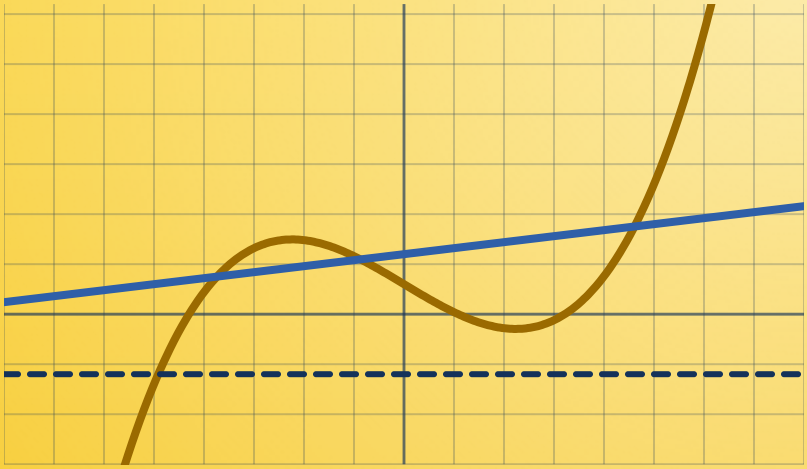
KECUDO

2026 EDITION · COLLEGE MATH SERIES



REAL ANALYSIS

Class 13

Differentiation

Slopes, proved from first principles

PREMIUM EDITION · KECUDO · 13

13 CHAPTERS · 120 SELF-ASSESSMENT QUESTIONS

INTERACTIVE EDITION + PRINT PDF

Contents

Part I · Foundations 4

1. The Derivative, Rigorously 5
2. Rules Proved 7
3. Carathéodory's Theorem 9
4. Interior Extremum Theorem 11

Part I Review · 10 questions **13**

Part II · Core 15

5. Rolle's Theorem 16
6. The Mean Value Theorem 18
7. Darboux's Theorem 20
8. L'Hôpital's Rule Proved 22
9. Taylor's Theorem 24

Part II Review · 10 questions **26**

Part III · Advanced 28

10. Power Series 29
11. Differentiating Series Term by Term 31
12. A Continuous, Nowhere-Differentiable Function 33
13. The Inverse Function Theorem 35

Part III Review · 10 questions **37**

Final Exam · 25 questions **39**

Your Results **44**

FOUNDATION

Chapter 1

The Derivative, Rigorously

$$f'(c) = \lim_{x \to c} (f(x) - f(c))/(x - c), \text{ when this limit exists}$$

WORKED EXAMPLE **f(x) = x² at c**

1. $(x^2 - c^2)/(x - c) = x + c \to 2c$.

✓ **f′(c) = 2c**

DIFFERENTIABLE ⇒ CONTINUOUS

$f(x) - f(c) = [(f(x) - f(c))/(x - c)] \cdot (x - c) \to f'(c) \cdot 0 = 0$. The converse is false.

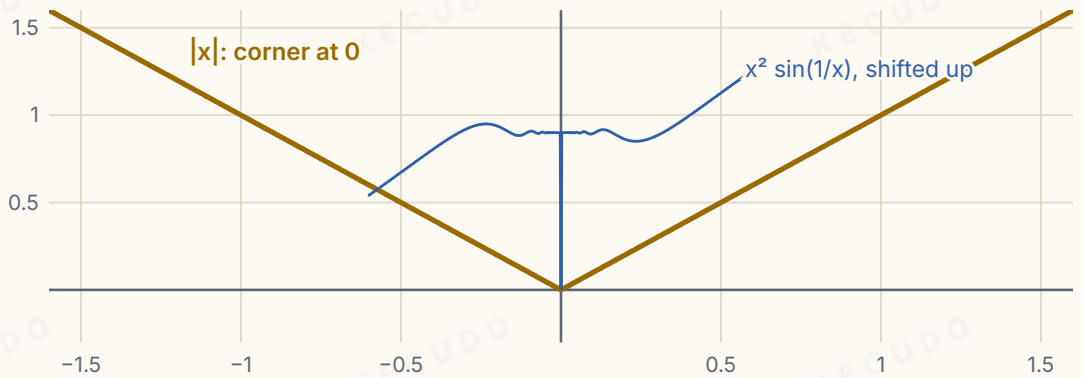


FIG 1.1 $|x|$ is continuous but has a corner at 0 (left slope −1, right slope 1). $x^2 \sin(1/x)$ wiggles infinitely often yet is differentiable at 0 with $f'(0) = 0$.

WORKED EXAMPLE **x² sin(1/x) (0 at 0) at 0**

1. $(f(x) - 0)/x = x \sin(1/x) \to 0$, so $f'(0) = 0$.
2. For $x \neq 0$: $f'(x) = 2x \sin(1/x) - \cos(1/x)$, which has no limit at 0.

✓ **A derivative can exist everywhere and still be discontinuous**

FOUNDATION

Chapter 2

Rules Proved

Rule	Statement	Key step in the proof
sum	$(f + g)' = f' + g'$	limit of a sum
product	$(fg)' = f'g + fg'$	add and subtract $f(x)g(c)$
quotient	$(f/g)' = (f'g - fg')/g^2$	$g(x) \to g(c) \neq 0$ by continuity
power	$(x^n)' = nx^{n-1}$	induction with the product rule

WORKED EXAMPLE **Product rule**

1. $f(x)g(x) - f(c)g(c) = f(x)[g(x) - g(c)] + g(c)[f(x) - f(c)]$.
2. Divide by $x - c$ and let $x \to c$; $f(x) \to f(c)$ because f is continuous.

✓ **$(fg)'(c) = f(c)g'(c) + g(c)f'(c)$**

THE CHAIN RULE NEEDS CARE

The naive proof divides by $g(x) - g(c)$, which can be 0 for $x \neq c$. Carathéodory's theorem (next chapter) fixes this.

CHECK YOUR UNDERSTANDING

Chapter 1: The Derivative, Rigorously

5 questions · your answer locks when you choose it

1. The derivative of x^3 at 2 (from the definition):

- **A** 12
- **B** 8
- **C** 6
- **D** 3

2. Differentiable at c implies:

- **A** nothing
- **B** continuous at c
- **C** f is a polynomial
- **D** f′ continuous

3. $|x|$ at 0 is:

- **A** continuous, not differentiable
- **B** discontinuous
- **C** undefined
- **D** differentiable

4. $x^2 \sin(1/x)$ (0 at 0): $f'(0)$ =

- **A** 1
- **B** 0
- **C** does not exist
- **D** −1

5. Is f′ always continuous when it exists everywhere?

- **A** Only for polynomials
- **B** Only on [a, b]
- **C** No
- **D** Yes

FINAL EXAM

25 Questions on the Whole Book

Parts A–E · 5 questions each

Part A · Basics

1. f′(c) is:

A the area under f
B lim (f(x) − f(c))/(x − c)
C f(c)/c
D lim f(x)

2. Differentiable ⇒

A monotone
B continuous
C bounded on ℝ
D polynomial

3. Rolle's Theorem requires f(a) =

A 0
B a
C f(b)
D f′(b)

4. Darboux: derivatives satisfy the:

A EVT
B Heine–Borel theorem
C intermediate value property
D M-test

5. Taylor's remainder in Lagrange form uses:

A f(c)
B $f^{(n+1)}(c)$
C the integral
D f′(a) only



KECUDO

Thank you for learning with Kecudo.

NEXT IN THE SERIES

Class I4 · Riemann Integration